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A General Approach to Edge Singularity Extraction Near Composed Wedges in Boundary-Element Method

Pierluigi Cecchini, Fernando Bardati, and Rodolfo Ravanelli

***Abstract*—A general approach, based on the two-dimensional boundary-element method (BEM), has been proposed to extract the electromagnetic-field singularities in the presence of composed wedges, i.e., those formed by adjacent dielectric and conducting bodies. The method requires the knowledge of the field singularity order and is based on solution factorization into both a regular part and a singular one. Only the regular part has to be determined after extraction. No restrictions are imposed on position and order of singularities since each edge is treated independently of the others. Moreover, the method does not require the solution of further equations or use of special basis functions. It naturally extends the conventional BEM approach, improving its accuracy and convergence performances. Examples are given for a microstrip transmission line with a strip of finite thickness. The results show practicability and advantages of the new approach.**

***Index Terms*—BEM, edge singularity.**

I. Introduction

The electromagnetic-field behavior near composed wedges, formed by adjacent dielectric and conducting bodies, has been widely investigated by several authors. Adjacent wedges of different dielectric materials, as well as a conducting one, having the tip in common, compose the angular domain. Static solutions for this problem can be found in [1]–[6]. Meixner investigated the time-varying case [1] and postulated that the field near an edge can be locally expressed as a series, whose first term takes the singular behavior into account. In his analysis, the results for the static case are presented as the zero-frequency limit of the dynamic one. The static solution characterizes the local field behavior even in a time-varying case, which can be imagined as the quasi-static limit in a region whose dimensions, compared with the wavelength, are small. Andersen and Solodukhiov observed that the Meixner series is not self-consistent in the case where all wedges are dielectric and their angles are a fractional multiple of π [2]; the singularity order is still determined by the dominant static term. In [3], a characteristic equation was obtained for the singularity order in the general case of N dielectric wedges and a perfectly conducting one.

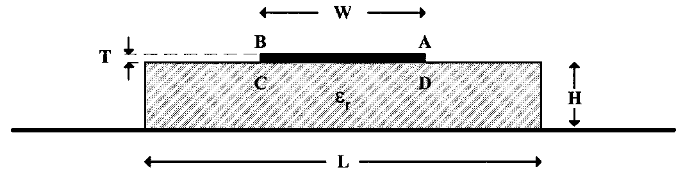

Fig. 1. Cross section of microstrip transmission line with truncated dielectric and finite thickness strip ($W = 4$ mm, $H = 2$ mm, $L = 24$ mm, $T = 0.2$ mm, $\varepsilon_r = 5$).

It is well known that the field singularities can be modeled for numerical computations in order to speed up convergence and reduce memory. In fact, direct numerical solutions to such problems require large numbers of basis functions and well-refined domain discretization [7], [8]. Alternatively, solutions can be achieved incorporating suitable edge-expansion functions into the numerical schemes [5], [9]–[13]. In problems that can be modeled by an integral equation over the interval $[-L, L]$, the edge condition can be expressed by means of a suitable entire-domain expansion

$$F(x) = \frac{1}{\left[1 - (x/L)^2\right]^{\alpha}} \sum_{n=0}^{\infty} c_n P_n^{\alpha}\left(\frac{x}{L}\right)$$

where $F(x)$ represents a singular field component, $\alpha \in (0, 1)$ is the order of the singularity as a function of the distance ρ from the tip (i.e., the field behaves as $1/\rho^{\alpha}$), and $P_n^{\alpha}(x/L)$ are orthogonal polynomials over $[-L, L]$ with respect to the weighting function $[1 - (x/L)^2]^{\alpha}$, such as Chebyshev ($\alpha = 0.5$) or Gegenbauer polynomials. This approach properly works when the unknown function exhibits a symmetrical singular behavior at the end points of the interval.

On the other hand, sub-domain edge functions can only be used near edges, with the advantage that each edge can be treated in a different way. However, additional work may be necessary to implement different kinds of basis functions over each sub-domain and to link the solutions [5], [14]. A further technique, based upon the boundary integral equation [9], consists in approximating the field near an edge by the first N terms of the Meixner series, with unknown coefficients. The resulting linear problem has more unknowns than equations and further approximate conditions ($N - 1$ per edge) have to be imposed at suitable points away from the edge. In [9], an indirect BEM approach to handle square-root edge singularities has been presented with reference to microstrips. It is based on the extraction of factor $\sqrt{1 - (2x/w)^2}$ from the charge density over the strip (of width w) and leads to a linear system involving the remaining regular charge density factor.

In this paper, we generalize the last approach, developing a technique, suitable for BEM, to extract the field singularities, whose order has already been determined in the proximity of composed wedges. The method is able to solve problems with singularities of general order and position; it does not require the solution of additional equations and makes use of the same polynomial basis for both the composed wedge and regular boundary. Therefore, the present procedure can be easily implemented as an extension of a standard BEM code. It can

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P. Cecchini and F. Bardati are with the Dipartimento di Informatica, Universita Roma Tor Vegata, 00133 Rome, Italy.
R. Ravanelli is with the Alenia Aerospazio, 00131 Rome, Italy.
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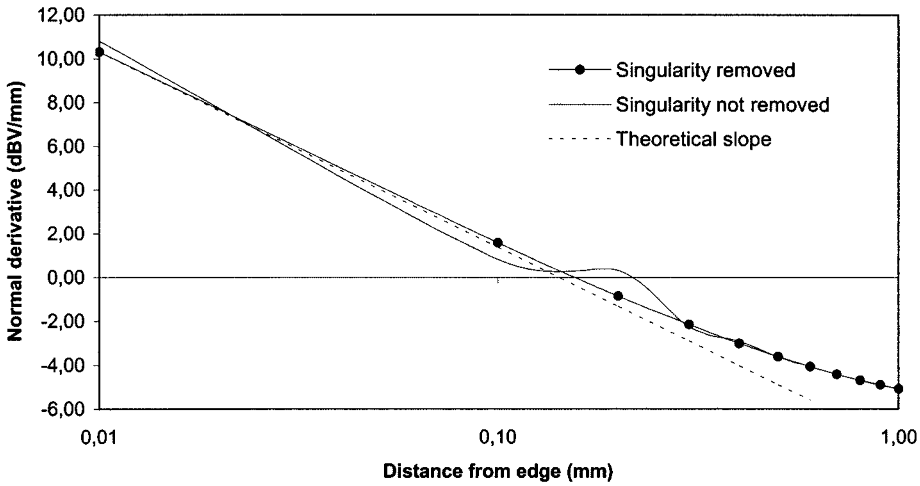

Fig. 2. Potential function normal derivative on line CD of Fig. 1 (in decibels per volts/millimeter) versus distance from D.

be used for static analysis of TEM and quasi-TEM modes in transmission lines, computation of transmission-line parameters, and solution of plane-wave edge-scattering problems.

II. Analytical Formulation

The standard BEM formulation for a two-dimensional Helmholtz or Laplace equation in an inhomogeneous domain can be found in [15]–[20], thus, it will not be repeated here. Let us assume that the closed boundary of each homogeneous region can be divided into M regular arcs Γ_l, whose parametric representations are $\underline{r}_l(t) = [x_l(t), y_l(t)]$, for $t \in [t_l, t_{l+1}]$, $l = 1, \ldots, M$. At a boundary point $\underline{r}_i$, the following integral equation can be written:

$$c_i \Phi_i = \sum_{l=1}^{M} \int_{t_l}^{t_{l+1}} G_l(t; \underline{r}_i) \frac{\partial \Phi_l(t)}{\partial n_l} |\underline{r}_l'(t)| dt - \sum_{l=1}^{M} \int_{t_l}^{t_{l+1}} \Phi_l(t) \frac{\partial G_l(t; \underline{r}_i)}{\partial n_l} |\underline{r}_l'(t)| dt \quad (1)$$

Here, $\underline{r}_l'(t) = d\underline{r}_l/dt$. $\Phi_l(t) = \Phi(\underline{r}_l(t))$, and $G_l(\underline{r}_i; t) = G(\underline{r}_i; \underline{r}_l(t))$ denote potential and Green's functions, respectively, at $\underline{r}_1(t)$. c_i is residual coefficient (0.5 at all boundary regular points), Φ_i is the potential function at $\underline{r}_i$, and $\underline{\hat{n}}_l$ is outward normal to Γ_l. The normal derivative of $\Phi_l(t)$, which will be indicated by $\Psi_l(t)$, is to be determined on the arcs, where a Dirichlet condition holds, and on interfaces between different media. Depending on the nature of the electromagnetic problem, $\Psi_l(t)$ is proportional to unknown electric or magnetic charge or current density. It may diverge at some points in field problems with edge singularities. The order of the singularities can be determined from Meixner series [1]. In a small circle with its center on the tip and radius r (with $r \ll \lambda$), the behavior of $\Psi_l(t)$ can be approximated by the dominant term behaving as $1/\rho^\alpha$, where ρ is distance from the tip and $0 < \alpha < 1$ depends on the local geometrical configuration and materials.

We choose the boundary arcs to contain one singularity of $\Psi_l(t)$, at the most, at either end point. As a consequence, the number of arcs M is not necessarily the minimum one. For simplicity of notation, we assume that the first M_s arcs Γ_l ($l = 1, \ldots, M_s$) contain an edge singularity, while the remaining M_{ns} do not, with $M = M_s + M_{ns}$.

This paper essentially deals with the efficient evaluation of the integrals in (1) when they are defined on curves with singularities ($l \leq M_s$). We restrict ourselves to the neighborhood of points for which there is a well-defined tangent along the edge. According to [1], we then consider the edge as locally straight. Therefore, let Γ_l ($l \leq M_s$) be an oriented straight-line segment at an angle θ_0 with the x-axis. The parametric equations of Γ_l can be written for $t \in [-1, 1]$ as

$$x(t) = x_0 + t\frac{L}{2}\cos(\theta_0) \quad y(t) = y_0 + t\frac{L}{2}\sin(\theta_0) \quad (2)$$

where L is length and (x_0, y_0) are midpoint coordinates. Assume that $\Psi_l(t)$ is singular at $t = -1$, with order α^-. The distance between $\underline{r}(t) = [x(t), y(t)]$ and the end point $t = -1$ can be written as

$$\rho(t) = \frac{L}{2}(1 + t), \qquad -1 \leq t \leq 1 \quad (3)$$

We then define a new function $\hat{\Psi}_l(t)$ as the *regular part* of $\Psi_l(t)$ according to

$$\hat{\Psi}(t) = \Psi(t)\left(\frac{\rho(t)}{L}\right)^{\alpha^-} = \Psi(t)\left(\frac{1+t}{2}\right)^{\alpha^-}, \qquad -1 < t \leq 1 \quad (4)$$

$\hat{\Psi}_l(t)$ is nonsingular, end points included, therefore, it can be easily approximated by means of polynomials. As an advantage of the proposed regularization, Ψ_l is equal to Ψ_m at the nonsingular common end-point of two adjacent arcs Γ_l and Γ_m. Substitution of (2) and (4) into (1) yields the following equation for the integrals over arcs including edge-singularity contributions in $t = -1$:

$$\frac{L}{2}\int_{-1}^{1} G_l(\underline{r}_i; t)\Psi_l(t)dt = \frac{L}{2^{\beta^-}}\int_{-1}^{1} G_l(\underline{r}_i; t)\frac{\hat{\Psi}_l(t)}{(1+t)^{\alpha^-}}dt \quad (5)$$

where $\beta^- = 1 - \alpha^-$. By introducing a new variable z as

$$z = (1+t)^{\beta^-}, \qquad -1 < t \leq 1 \quad (6)$$

the integral on the right-hand side of (5) becomes

$$\frac{L}{2^{\beta^-}\beta^-}\int_{0}^{2^{\beta^-}} G_l\left(\underline{r}_i; z^{1/\beta^-} - 1\right)\hat{\Psi}_l\left(z^{1/\beta^-} - 1\right)dz = \frac{L}{2^{\beta^-}\beta^-}\int_{0}^{2^{\beta^-}} G_l\left(\underline{r}_i; z\right)\hat{\Psi}_l(z)dz \quad (7)$$

where, for the sake of simplicity, we have used the same symbol for functions of t and z. If we now assume that, on Γ_l, the singularity of

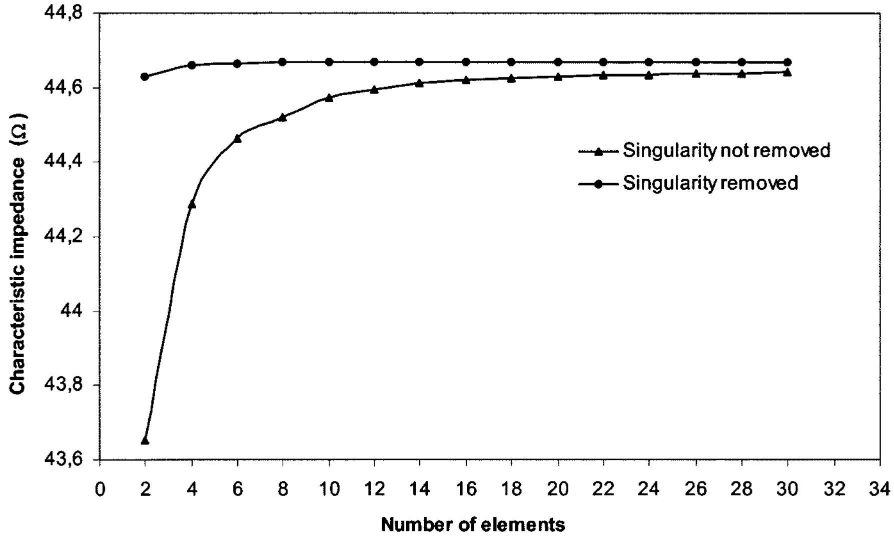

Fig. 3. Microstrip line characteristic impedance versus number of elements on line AB of Fig. 1.

$\Psi_l(t)$ is located at $t = +1$, with order α^+, it can easily be shown that the integral on the left-hand side of (5) can be rewritten as

$$\frac{L}{2^{\beta+}\beta+} \int_0^{2^{\beta+}} G_l\left(\underline{r}_i; 1 - z^{1/\beta^+}\right)\hat{\Psi}_l\left(1 - z^{1/\beta^+}\right)dz$$
$$= \frac{L}{2^{\beta+}\beta+} \int_0^{2^{\beta+}} G_l\left(\underline{r}_i; z\right)\hat{\Psi}_l(z)dz \quad (8)$$

with $\beta^+ = 1 - \alpha^+$. Thus, the original problem has been reformulated in terms of new unknown nonsingular functions $\hat{\Psi}_l(t)$. Equations (7) and (8) allow the integrals in (1) over boundary portions, including edge singularities, to be efficiently performed.

The above procedure can be advantageous for the evaluation of the following integral:

$$\frac{L}{2}\int_{-1}^{1} \Psi(t)dt \quad (9)$$

which is frequently met when an integral parameter, such the characteristic impedance, is the target of a BEM computation.

III. Numerical Results

The method of Section II has been applied to the quasi-static analysis of a microstrip line with truncated dielectric $\varepsilon_r\varepsilon_0$ and finite metallization thickness (Fig. 1). At edges A and B, the charge density singularity order is $\alpha = 1/3$. At C and D, two composed wedges are formed by the metallic strip and two dielectric media (air and substrate). In this case, it has been shown [1] that the singularity order α can be achieved from the solution of

$$\frac{\varepsilon_r - 1}{\varepsilon_r + 1} = \frac{\sin\left(\beta\frac{3\pi}{2}\right)}{\sin\left(\beta\frac{\pi}{2}\right)} \quad (10)$$

where $\alpha = 1 - \beta$ (the solution of (10) is explicit in [3]). We found 1/2.2386 for α $(\varepsilon_r = 5)$. It is worthy noting that the charge density has different orders of edge singularity on the short sides of the thick strip. In Fig. 2, the potential-function normal derivative (in decibels per volts/millimeters) versus distance from tip C is shown. At a small distance, the function is a straight line, whose slope gives the singularity order. Extracting the singularity according to the proposed approach provides a solution whose slope is in accordance with the theory, while it is affected by a 12% error if the singularity is not extracted. At some distance from the edge, higher order terms of the Meixner series are not negligible and solutions diverge from straight lines. To our knowledge, no theoretical solution is available for the direct comparison of the numerical results. However, the solution obtained after singularity extraction does not exhibit the oscillation, which instead affects the solution in the absence of extraction.

The diagrams in Fig. 3 refer to the evaluation of the characteristic impedance of the microstrip line of Fig. 1 for an increasing number of elements over the segment AB in the BEM scheme. A satisfactory convergence to the theoretical value $Zc = 44.63\ \Omega$ [21] can be appreciated with and without singularity extraction (accuracy better than 0.05%). However, the convergence rate is clearly improved if the edge singularities are removed by the proposed method.

The results have been obtained using quadratic elements, i.e., using second-order polynomials. The procedure has been implemented as an extension of an in-house C++ BEM code. The solution usually takes few seconds on a PC with 300-MHz 64-MB RAM. It may takes about 1 m when the number of elements increases, even if Fig. 3 shows that using many elements does not significantly improve the accuracy.

IV. Conclusions

In this paper, a general approach has been presented to cope with the singular behavior of the electromagnetic field near composed wedges in a two-dimensional BEM. It requires knowledge of the order of singularity for each such edge. Application to a microstrip transmission line shows that the proposed method improves the performance of a BEM computation in comparison with a conventional BEM scheme. In particular, it provides more accurate solutions for the field in the vicinity of the wedges and improves the convergence rate in the evaluation of integral parameters. It can easily be implemented to extend the performance of a standard BEM code.

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